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Some Special Integer Partitions Generated by a Family of Functions

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ABSTRACT. In this work, inspired by Ramanujan's fifth order Mock Theta function $f_1(q)$, we define a collection of functions and look at them as generating functions for partitions of some integer n containing at least m parts equal to each one of the numbers from 1 to its greatest part s, with no gaps. We set a two-line matrix representation for these partitions for any $m \geq 2$ and collect the values of the sum of the entries in the second line of those matrices. These sums contain information about some parts of the partitions, which lead us to closed formulas for the number of partitions generated by our functions, and partition identities involving other simpler and well known partition functions.

Keywords: integer partition, mock theta function, matrix representation, partition identity.

1 INTRODUCTION

Mock Theta functions were introduced by Ramanujan shortly before his early death in a letter sent to Hardy, in 1920. At that time, what Ramanujan meant for a mock theta function was not very clear [8]. However, nowadays these functions have been largely explored and many applications, as for example in modular forms, have appeared [14, 18, 19, 20]. Great historical backgrounds concerning Ramanujan's life and work and its range along time can be found in [2, 3, 4, 5, 6], to name a few classical references (a nice and easy reading is [16] by Ono).

Besides modular forms, mock theta functions have an interesting interpretation when seen as generating functions for integer partitions, defined as follows.

Definition 1.1. *Given $n \in \mathbb{Z}_+$, a* partition *of n is a list $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, with $\lambda_t \geq \lambda_{t+1}$ for all $1 \leq t \leq s-1$, such that $\sum_{t=1}^{s} \lambda_t = n$. Also, each λ_t is called a* part *of the partition.*

Remark 1.2. *A partition of n as described above may also be called an* unrestricted partition of *n (or a* regular partition of *n) since there's no restriction to the number or size of its parts λ_t.*

Remark 1.3. *Although the usual notation for a partition of an integer n is the one given in Definition 1.1, in our demonstrations along this work the order of the parts will not be necessarily*

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preserved for purposes of simplification. Also, we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ *and* $\sum_{t=1}^{s} \lambda_t = n$ *both with the same meaning.*

Definition 1.4. *Let* $P(n)$ *denote the set of partitions of an integer n. We write* $p(n)$ *to denote the number of elements of* $P(n)$*, that is, the number of partitions of n. Then* $|P(n)| = p(n)$ *and* $p(n)$ *is called the* partition function. *Whenever the parts of the partitions we are interested in have a restriction, it is usual to write* $p(n, \text{“}restriction\text{”})$.

The partition function $p(n)$ has the well known generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

However, the values of $p(n)$ can also be seen as coefficients of Ramanujan's third order mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

The coefficients of $f(q)$ are related to Dyson's rank of a partition, defined in [15], which motivated a conjecture by Andrews [1] on the values of these coefficients.

Proved in 2006 by Bringmann and Ono [12], Andrews' conjecture turned out to be true. Not only the coefficients of mock theta function $f(q)$ were determined, but also a new formula for $p(n)$ became known. Latterly the formula for $p(n)$ was greatly improved by Bruinier and Ono [13], showing it can be written as a finite sum of algebraic numbers.

In a work of 2013 [11], Brietzke et al. presented a combinatorial interpretation as two-line matrices for many mock theta functions, and consequently for many different types of integer partitions (see Table in [11], page 240). A few years before, Santos et al. in [17] gave three distinct matrix representations for unrestricted partitions, one of them completely describing the conjugate partition. The bijective proofs between the set of partitions and the set of two-line matrices can be found in [17] and [10].

Motivated by these ideas, the present work is inspired by a matrix representation for the mock theta function $f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}$, described in [11], page 241. This function counts partitions and considers weights -1 and $+1$ for each one of them. Here we consider what we have called the unsigned version of $f_1(q)$, that is, function

$$f_1^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n},$$

which only eliminates the different weights from $f_1(q)$ by removing the negative sign from $(-q;q)_n$.[1]

[1]Expressions $(q;q)_n$ and $(-q;q)_n$ come from the usual notation given by

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k)$$

We enunciate the matrix representation for $f_1^*(q)$ as the following theorem.

Theorem 1.5. *The coefficient of q^n in the expansion of $f_1^*(q)$ is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1.1}$$

with non-negative integer entries satisfying $c_s = 2$, $c_t = 2 + c_{t+1} + d_{t+1}$, $\forall t < s$, and $n = \sum c_t + \sum d_t$.

When seen as generating function for integer partitions, $f_1^*(q)$ counts the partitions of n containing all parts from 1 to some s, with no gaps, and multiplicity at least two. This means that the number of partitions of n counted by $f_1^*(q)$ equals the number of matrices of type (1.1) described in Theorem 1.5.

Partition identities and closed formulas concerning $f_1^*(q)$ and some other functions can be found in [9].

Now we define a collection of functions inspired by the definition of $f_1^*(q)$. We call these functions $f_*^m(q)$, with fixed $m \geq 2$, and write them as

$$f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}. \tag{1.2}$$

For a fixed $m \geq 2$, the general term

$$\frac{q^{m(1+2+3+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}$$

generates the partitions for some integer containing at least m parts equal to each one of the numbers $1,2,3,\ldots,s$, with no gaps. By conjugation, this general term also generates the partitions for some integer into exactly s parts, with smallest part $\lambda_s \geq m$ and with difference between consecutive parts $\lambda_t - \lambda_{t+1} \geq m$, for any $t < s$.

Remark 1.6. *Function $f_*^m(q)$ with $m = 2$ is the same as function $f_1^*(q)$. In the present work we deal with general aspects of function $f_*^m(q)$, for any $m \geq 2$. For more specific details about $f_*^2(q) = f_1^*(q)$ see [9].*

In the following pages we present the matrix representation for integer partitions counted by $f_*^m(q)$ and a collection of results derived from this representation, concerning the integer partitions given by the generating functions $f_*^m(q)$.

2 THE FAMILY $(f_*^m(q))_{m\geq 1}$

Let us consider a partition of n counted by the function $f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}$, for some fixed $m \geq 2$. Recall that such a partition has each part from 1 to s with no gaps and multiplicity at least m. So, n can be written as

$$n = (m+d_s)\cdot s + (m+d_{s-1})\cdot(s-1) + \cdots + (m+d_2)\cdot 2 + (m+d_1)\cdot 1,$$

with d_t a non negative integer for all $1 \leq t \leq s$.

By rearranging these numbers, we may have n as the sum of the entries of the matrix

$$A = \begin{pmatrix} s\cdot m + d_2 + d_3 + \cdots + d_s & (s-1)\cdot m + d_3 + \cdots + d_s & \cdots & 2m + d_s & m \\ d_1 & d_2 & \cdots & d_{s-1} & d_s \end{pmatrix}, \tag{2.1}$$

with d_t a non negative integer for all $1 \leq t \leq s$, which allows us to state the following theorem.

Theorem 2.1. *The coefficient of q^n in the expansion of $f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}$ equals the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2.2}$$

with non-negative integer entries, satisfying $c_s = m$, $c_t = m + c_{t+1} + d_{t+1}$, $\forall t < s$, and $n = \sum c_t + \sum d_t$.

Given any partition generated by function $f_*^m(q)$, the second row of its associated matrix informs us how many parts, besides the m copies of each part from 1 to s, the partition has. This then motivates the following definition.

Definition 2.2. *Let $P_{[s]}^{m[s]}(n,k)$ be the set of partitions of n into parts ranging from 1 to s, with no gaps and multiplicity m, and k other parts from 1 to s. Also, let $p_{[s]}^{m[s]}(n,k)$ denote the cardinality of $P_{[s]}^{m[s]}(n,k)$.*

Remark 2.3. *We use the notation $[s] := \{n \in \mathbb{N} \mid 1 \leq n \leq s\} = \{1,2,3,\ldots,s-1,s\}$ referring to the range of parts of a partition counted by $f_*^m(q)$. This notation appears again in Section 5.*

Motivated by Definition 2.2, given a fixed m, for each n we classify its partitions according to the sum of the entries in the second row of the associated matrix. For different values of m, we count the appearance of each number in these sums and organize the data on tables.

In any of those tables the entry in line n and column $n-k$ is the number of times k appears as sum of the entries of the second row in type (2.2) matrices. That is, how many partitions of n have k extra parts, besides the m copies of each integer from 1 to s. Excerpts of the tables obtained for $m = 4$ and 5 are presented below as examples (Tables 1 and 2).

Tables 1 and 2 and other ones that can be obtained for other values of m, which we omit from this text, have interesting values of $p_{[s]}^{m[s]}(n,k)$, for different k. In order to refer to these values in a simpler way, we adopt the following definition.

Definition 2.4. *Given $m \geq 2$ and $k \geq 0$ we call the sequence of $p_{[s]}^{m[s]}(n,k)$ for $n \geq 0$ the kth diagonal of the associated table, built according to the description above.*

Table 1: Values of $p_{[s]}^{4[s]}(n,k)$ in line n and column $n-k$, that is, number of partitions of n having k extra parts, besides 4 copies of each integer from 1 to s.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
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4	0	0	0	0	1																																									
5	0	0	0	0	1	0																																								
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9	0	0	0	0	1	0	0	0	0	0																																				
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16	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0																													
17	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0	0																												
18	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0	0																											
19	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0																										
20	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0																									
21	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0																								
22	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0																							
23	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0																						
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27	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	0																		
28	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	0	0																	
29	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	1	0	0																
30	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	1	0	0															
31	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	2	0	0	0														
32	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	1	0	0	0													
33	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	2	1	0	0	0												
34	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	2	0	0	0	0											
35	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	3	1	0	0	0	0										
36	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	2	1	0	0	0	0									
37	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	3	2	0	0	0	0	0								
38	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	4	3	1	0	0	0	0	0							
39	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	4	4	2	1	0	0	0	0	0						
40	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	3	2	0	0	0	0	0	1					
41	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	4	3	1	0	0	0	0	1	0				
42	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	4	2	1	0	0	0	1	1	0			
43	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	5	3	2	0	0	0	1	1	1	0		
44	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	6	4	3	1	0	0	1	1	2	1	0	
45	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	6	5	4	2	1	0	1	1	2	2	0	0

Table 2: Values of $p_{[s]}^{5[s]}(n,k)$ in line n and column $n-k$, that is, number of partitions of n having k extra parts, besides 5 copies of each integer from 1 to s.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
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7	0	0	0	0	0	1	0	0																																						
8	0	0	0	0	0	1	0	0	0																																					
9	0	0	0	0	0	1	0	0	0	0																																				
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12	0	0	0	0	0	1	0	0	0	0	0	0	0																																	
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14	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0																															
15	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1																														
16	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0																													
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18	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0																											
19	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0																										
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26	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0																			
27	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0																		
28	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0																	
29	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0																
30	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1															
31	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	0														
32	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0													
33	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	0												
34	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	2	0	0											
35	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	1	0	0										
36	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	2	1	0	0									
37	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	2	2	0	0	0								
38	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	2	3	1	0	0	0							
39	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	3	2	1	0	0	0						
40	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	3	3	2	0	0	0	0					
41	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	3	3	1	0	0	0	0				
42	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	3	4	2	1	0	0	0	0			
43	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	4	3	2	0	0	0	0	0		
44	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	3	1	0	0	0	0	0	
45	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	4	4	2	1	0	0	0	0	0

Remark 2.5.

(i) *Note that the kth diagonal makes sense only for $n \geq k$. So, from now on we omit $p_{[s]}^{m[s]}(0,k)$ for any $k \geq 1$ and assume $p_{[s]}^{m[s]}(n,k) = 0$ for $1 \leq n < k$ ($p_{[s]}^{m[s]}(0,0) = 1$, by definition).*

(ii) *Moreover, the results for any kth diagonal are valid for functions $f_*^m(q)$ whenever $m \geq k$.*

Some facts about the zero and the first diagonals are easily observed and set below.

Definition 2.6. *We define $T_j := \dfrac{j(j+1)}{2}$, the jth triangular number for j a positive integer.*

Proposition 2.7. *Given $m \geq 2$, we have*

(i) $$p_{[s]}^{m[s]}(n,0) = \begin{cases} 1, & \text{if } n = m \cdot T_j, \text{ for some } j \geq 0; \\ 0, & \text{otherwise} \end{cases}$$

and

(ii) $$p_{[s]}^{m[s]}(n,1) = \begin{cases} 1, & \text{if } n = m \cdot T_j + i, \text{ for some } j \geq 1 \text{ and } 1 \leq i \leq j; \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

(i) Observe that any triangular number $T_j = \dfrac{j(j+1)}{2}$ can also be written as $T_j = 1 + 2 + 3 + \cdots + j$. Therefore, $n = m \cdot T_j = m(1 + 2 + 3 + \cdots + j)$ and $\lambda = (\underbrace{1,1,\ldots,1}_{m},\underbrace{2,2,\ldots,2}_{m},\underbrace{3,3,\ldots,3}_{m},\ldots,\underbrace{j,j,\ldots,j}_{m})$ is precisely the only partition of $n = m \cdot T_j$ with parts ranging from 1 to some s, with no gaps and multiplicity exactly m. Therefore, $p_{[s]}^{m[s]}(n,0) = 1$ if $n = m \cdot T_j$ for any $j \geq 0$.

Any other n satisfying $m \cdot T_j < n < m \cdot T_{j+1}$ is such that $n = m(1+2+3+\cdots+j)+k$, with $k < m \cdot (j+1)$. Then, in order for a partition of n to have m parts ranging from 1 to some s with no gaps, it has to have also some other part in $[s]$. Therefore, $p_{[s]}^{m[s]}(n,0) = 0$ if n is not m times a triangular number.

(ii) If $n = m \cdot T_j + i = m(1 + 2 + 3 + \cdots + j) + i$, note that $\lambda = (\underbrace{1,1,\ldots,1}_{m},\underbrace{2,2,\ldots,2}_{m},\underbrace{3,3,\ldots,3}_{m},\ldots,\underbrace{j,j,\ldots,j}_{m},i)$ is precisely the only partition of $n = m \cdot T_j + i$ with parts ranging from 1 to some s, with no gaps and multiplicity m and only one extra part i of size $1 \leq i \leq j$.

Although i could be itself partitioned into more than one part, that would not be a partition counted by $p_{[s]}^{m[s]}(n,1)$. Neither would it be if $j < i < m \cdot (j+1)$. Therefore, this leads to $p_{[s]}^{m[s]}(n,1) = 0$ if $n \neq m \cdot T_j + i$ for any triangular number T_j and $1 \leq i \leq j$, and equality (ii) holds.

□

In fact, Proposition 2.7 gives us a complete characterization of the zero and the first diagonals of $f_*^m(q) = \sum_{n=0}^{\infty} \dfrac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}$ table, for any value of m.

Similar results were obtained in order to characterize other diagonals of the tables. In the next sections we show the sequence of values of those diagonals for $1 \leq n \leq 200$, clearly many more than those shown in Tables 1 and 2, and present the results derived from the analysis of these sequences.

3 THE 2nd DIAGONAL

The table below (Table 3) shows us the sequence of values contained in the 2nd diagonal, that is, the values of $p_{[s]}^{m[s]}(n,2)$ for $1 \leq n \leq 200$ counted by functions $f_*^2(q)$, $f_*^3(q)$, and $f_*^4(q)$.

Table 3: Values contained in the 2nd diagonals of the tables of functions $f_*^2(q)$, $f_*^3(q)$, and $f_*^4(q)$.

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,2)$
$f_*^2(q)$	$(0,0,0,1,0,0,0,1,1,1,0,0,0,1,1,2,1,1,0,0,0,1,1,2,2,2,1,1,0,0,0,1,1,2,2,3,2,2,1,1,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,6,5,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,6,6,6,5,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,6,6,7,6,6,5,5,\ldots)$
$f_*^3(q)$	$(0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,1,1,2,1,1,0,0,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,0,0,0,1,1,2,2,3,2,2,1,1,0,0,0,0,0,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,5,5,4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,\ldots)$
$f_*^4(q)$	$(0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,1,1,2,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,4,4,3,3,2,2,1,1,0,0,\ldots)$

Observe that each sequence of Table 3 has a lot of zeros in very regular positions. Also, between two lists of zeros, greater integers appear in such a way that each list of integers is symmetrical, always starting and ending at 1, and increasing and decreasing by at most 1 unit.

Moreover, note that each list of integers greater than zero has odd size. In some cases the central term is different from the others, while in other cases the central term is the same as its adjacent terms.

In the following results we formalize all of the previous observations, getting a complete characterization of the 2nd diagonal, for any n and any $m \geq 2$.

Proposition 3.1. *Given $m \geq 2$, for all $n \geq 1$ and $1 \leq i \leq (m-2)n+3$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2-i,2\right)=0.$$

Proof. Let us suppose we could partition $\frac{m(n^2+n)}{2}+2-i$ into m copies of each part from 1 to some j and two more parts less than or equal to j. So, we would write

$$\begin{aligned}\frac{m(n^2+n)}{2}+2-i &= \underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{2+\cdots+2}_{m}+\underbrace{1+\ldots+1}_{m}+r+s\\ &= \frac{m(j^2+j)}{2}+r+s,\end{aligned}$$

for some $j \geq 1$ and $1 \leq s \leq r \leq j$. Note that j has to be less than n, and so we can make the following estimates:

$$\frac{m(n^2+n)}{2}+2-i=\frac{m(j^2+j)}{2}+r+s \leq \frac{m((n-1)^2+(n-1))}{2}+2(n-1),$$

which is equivalent to

$$mn+2-i \leq 2n-2.$$

On the other hand,

$$mn+2-i \geq mn+2-(m-2)n-3,$$

and so

$$mn+2-(m-2)n-3 \leq 2n-2,$$

or

$$1 \leq 0,$$

which is absurd.

As we cannot write $\frac{m(n^2+n)}{2}+2-i=\underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{1+\ldots+1}_{m}+r+s$, this means

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2-i,2\right)=0.$$

□

The symmetry of the list of integers between the zeros is described in the next proposition. In order to prove it we need the following lemma, which will also be useful in further sections.

Lemma 3.2. *Given $m \geq k \geq 2$, for all $n \geq 2$ and $1 \leq t \leq n-1$ we have*

$$\frac{m(2nt - t^2 + t)}{2} > k(n-1). \tag{3.1}$$

Proof. First of all, as $m \geq k \geq 2$ let us write $m = k + j$, with $j \geq 0$. We prove inequality (3.1) by induction on $n \geq 2$. For $n = 2$ we have only $t = 1$, which implies

$$\frac{m(2nt - t^2 + t)}{2} = 2m = 2k + 2j > 2k - k = k \cdot n - k = k(n-1),$$

and so (3.1) is true.

By supposing (3.1) is true for some $n = b \geq 2$ and all $1 \leq t \leq b-1$, let us prove it also holds for $b+1$, with $1 \leq t \leq b$,

$$\frac{m(2(b+1)t - t^2 + t)}{2} = \frac{m(2bt - t^2 + t)}{2} + mt.$$

For $1 \leq t \leq b-1$, by induction hypothesis we have

$$\frac{m(2bt - t^2 + t)}{2} + mt > k(b-1) + mt \geq k(b-1) + k = k((b+1)-1).$$

For $t = b$ we have

$$\frac{m(2bt - t^2 + t)}{2} + mt \geq \frac{m(b^2 + b)}{2} + kb > kb = k((b+1)-1).$$

Therefore, by induction we have (3.1) valid for all $n \geq 2$ and $1 \leq t \leq n-1$. $\square$

The particular case of Lemma 3.2 with $k = 2$ is used in the following proposition.

Proposition 3.3. *Given $m \geq 2$, for all $n \geq 1$ and $0 \leq i \leq n-1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2n - i, 2\right).$$

Proof. We begin by claiming that the greatest part of any partition counted by the number $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right)$ is exactly n. Indeed, if the greatest part were larger than n, let us say $n+t$ with $t \geq 1$, we would have

$$m \cdot (n+t) + m \cdot (n+t-1) + \cdots + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i, \tag{3.2}$$

with $1 \leq s \leq r \leq n+t$. By doing some estimates and using Lemma 3.2 with $k = 2$ we get (3.2) equivalent to

$$r + s = 2 + i - \frac{m(2nt + t + t^2)}{2} \leq 0,$$

contradicting the fact that $1 \leq s \leq r$.

Moreover, for $n > 1$ the greatest part cannot be smaller than n either. Indeed, if it were $n-t$ with $t \geq 1$, we would have

$$m \cdot (n-t) + m \cdot (n-t-1) + \cdots + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i, \tag{3.3}$$

with $1 \leq s \leq r \leq n-t$. According to Lemma 3.2 with $k=2$, equation (3.3) is equivalent to

$$r + s = \frac{m(2nt - t^2 + t)}{2} + 2 + i > 2n - 2. \tag{3.4}$$

However, as $s \leq r \leq n-t$ we have $r+s \leq 2(n-t) \leq 2n-2$, and so inequality (3.4) is an absurd.

Therefore, the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2+i, 2\right)$ has to be n. Analogous arguments allow us to conclude that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2n-i, 2\right)$ is also exactly n.

Then, given λ a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2+i, 2\right)$, we have

$$\lambda = (\underbrace{n,\ldots,n}_{m},\ldots,\underbrace{2,\ldots,2}_{m},\underbrace{1,\ldots,1}_{m},r,s),$$

with $1 \leq s \leq r \leq n$. So,

$$m \cdot n + m \cdot (n-1) + \cdots + m \cdot 2 + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i,$$

and therefore

$$r + s = 2 + i.$$

By writing $\mu = (\underbrace{n,\ldots,n}_{m},\ldots,\underbrace{2,\ldots,2}_{m},\underbrace{1,\ldots,1}_{m},n+1-r,n+1-s)$ we get a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2n-i, 2\right)$ as

$$\begin{aligned} m \cdot n + \cdots + m \cdot 2 + m \cdot 1 + n + 1 - r + n + 1 - s &= \frac{m(n^2+n)}{2} + 2n + 2 - (r+s) \\ &= \frac{m(n^2+n)}{2} + 2n + 2 - (2+i) \\ &= \frac{m(n^2+n)}{2} + 2n - i. \end{aligned}$$

Easily we can build the reverse map, getting

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2+i, 2\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2n-i, 2\right).$$

□

We illustrate Proposition 3.3 by taking $m = 4$, $n = 9$, and $i = 7$ in the following example.

Example 3.4. *The number of partitions counted by* $p_{[s]}^{4[s]}(189,2)$ *is the same as* $p_{[s]}^{4[s]}(191,2)$*, as shown in Table 4.*

Table 4: Table for Example 3.17.

$P_{[s]}^{4[s]}(189,2)$	$P_{[s]}^{4[s]}(191,2)$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{8},\mathbf{1})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{9},\mathbf{2})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{7},\mathbf{2})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{8},\mathbf{3})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{6},\mathbf{3})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{7},\mathbf{4})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{5},\mathbf{4})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{6},\mathbf{5})$

In order to have the 2nd diagonal completely described, what remains to be shown is the exact value of each term in the 2nd diagonal of any function $f_*^m(q)$. This result is given next and has a simple demonstration.

Proposition 3.5. *Given* $m \geq 2$*, for all* $n \geq 1$ *and* $0 \leq i \leq n-1$ *we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1-i,2\right) = \left\lfloor \frac{n+1-i}{2} \right\rfloor.$$

Proof. By observing that $\frac{m(n^2+n)}{2}+2+i$ and $\frac{m(n^2+n)}{2}+n+1-i$ both range from 2 to $n+1$ if $0 \leq i \leq n-1$, Proposition 3.3 gives us that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1-i,2\right)$ is n. So we have

$$\frac{m(n^2+n)}{2}+n+1-i = m\cdot n+\cdots+m\cdot 2+m\cdot 1+r+s,$$

for $1 \leq s \leq r \leq n$, which can be rewritten as

$$r+s = n+1-i. \tag{3.5}$$

According to [7], the number of partitions of $n+1-i$ into exactly two parts, that is, the number of solutions of (3.5) satisfying $1 \leq s \leq r \leq n$ is $\left\lfloor \frac{n+1-i}{2} \right\rfloor$. □

Now we may observe that, as it happens with $\frac{m(n^2+n)}{2}+2+i$ and $\frac{m(n^2+n)}{2}+n+1-i$, also $\frac{m(n^2+n)}{2}+2n-i$ and $\frac{m(n^2+n)}{2}+n+1+i$ both range through the same values for $0 \leq i \leq n-1$. Therefore, by joining Propositions 3.3 and 3.5 we get the following corollary.

Corollary 3.6. *Given $m \geq 2$, for all $n \geq 1$ and $0 \leq i \leq n-1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1\pm i, 2\right) = \left\lfloor \frac{n+1-i}{2} \right\rfloor.$$

Example 3.7. *For $m = 4$, $n = 6$ and $0 \leq i \leq 5$ we have the partitions shown in Table 5.*

Table 5: Table for Example 3.20.

$91 \pm i$	$P_{[s]}^{4[s]}(91 \pm i, 2)$	$p_{[s]}^{4[s]}(91 \pm i, 2)$
86	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{1},\mathbf{1})$	1
87	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{2},\mathbf{1})$	1
88	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{2},\mathbf{2})$	2
89	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{2})$	2
90	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{3})$	3
91	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{3})$	3
92	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{3})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{4})$	3
93	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{3})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{4})$	2
94	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{4})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{5})$	2
95	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{5})$	1
96	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{6})$	1

Now we have the 2nd diagonal of function $f_*^m(q)$ table, for any $m \geq 2$, completely described.

4 THE 3rd DIAGONAL

The sequences of values of $p_{[s]}^{m[s]}(n,3)$ for $1 \leq n \leq 200$, contained in the 3rd diagonal of the tables of functions $f_*^3(q)$, $f_*^4(q)$, and $f_*^5(q)$, are shown in Table 6 below.

The zeros in any of the sequences of Table 6 are described in the next result. Its proof is analogous to the proof of Proposition 3.1 and, therefore, is omitted.

Table 6: Values contained in the 3rd diagonals of the tables of functions $f_*^3(q)$, $f_*^4(q)$, and $f_*^5(q)$.

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,3)$
$f_*^3(q)$	(0,0,0,0,0,1,0,0,0,0,0,1,1,1,1,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,1,1,2,3,3,3, 3,2,1,1,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,6,6,6,6,5, 4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,7,8,8,8,7,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3, 4,5,7,8,9,10,10,10,10,9,8,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,8,10,11,12, 12,13,12,12,11,10,8,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,8,10,12,13,14,15, 15,15,15,14,13,12,10,8,7,5,4,3,2,1,1,0,0,0,0,0,...)
$f_*^4(q)$	(0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0, 0,0,0,0,1,1,2,3,3,3,3,2,1,1,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0, 0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,6,6,6,6,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0, 1,1,2,3,4,5,7,7,8,8,8,7,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5, 7,8,9,10,10,10,10,9,8,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5, 7,8,10,11,12,12,13,12,12,11,10,8,...)
$f_*^5(q)$	(0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,2,1, 1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,3,3,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0, 0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,6, 6,6,6,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,7,7,8,8, 8,7,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,7,8, 9,10,10,10,10,9,8,7,5,4,...)

Proposition 4.1. *Given $m \geq 3$, for all $n \geq 1$ and $1 \leq i \leq (m-3)n+5$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+3-i,3\right)=0.$$

Also, non-zero values are again finite symmetrical lists of integers, which get longer as n increases. This leads us to the following result.

Proposition 4.2. *Given $m \geq 3$, for all $n \geq 1$ and $0 \leq i \leq \left\lfloor \frac{3n-1}{2} \right\rfloor - 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+3+i,3\right)=p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+3n-i,3\right).$$

The proof of Proposition 4.2 follows the same direction as the proof of Proposition 3.3, by building an analogous bijection and using the particular case of Lemma 3.2 with $k=3$.

Example 4.3. *For $m=3$, $n=5$ and $i=4$ we have the same number of elements in $P_{[s]}^{3[s]}(52,3)$ as in $P_{[s]}^{3[s]}(56,3)$, as shown in Table 7.*

Table 7: Table for Example 4.23.

$P_{[s]}^{3[s]}(52,3)$	$P_{[s]}^{3[s]}(56,3)$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{1},\mathbf{5},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{2},\mathbf{4},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{4},\mathbf{4})$

The values described in Proposition 4.2 compose a sequence which has a combinatorial interpretation in terms of another type of partitions, easier to count. This follows in the theorem below.

Theorem 4.4. *Given $m \geq 3$, for all $n \geq 1$ and $j \geq 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+2,3\right) = p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+n+2,3\right)$$
$$= p(n-1, \textit{at most } 3 \textit{ parts}).$$

Proof. First of all, note that $\frac{m(n^2+n)}{2} + n + 2 = \frac{m(n^2+n)}{2} + 3 + (n-1)$. Since $0 \leq n-1 \leq \left\lfloor \frac{3n-1}{2} \right\rfloor - 1$, we already know from Proposition 4.2 that a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+2,3\right)$ has n as its greatest part. So, by writing

$$m\cdot n+\cdots+m\cdot 2+m\cdot 1+r+s+t = \frac{m(n^2+n)}{2}+n+2,$$

with $1 \leq t \leq s \leq r \leq n$, we get

$$r+s+t = n+2.$$

By decreasing r, s and t in one unit we get $r' = r-1$, $s' = s-1$ and $t' = t-1$. Therefore $r'+s'+t' = r+s+t-3 = n-1$, with $0 \leq t' \leq s' \leq r' \leq n-1$. So, (r',s',t') is a triple of non-negative integers that add up to $n-1$. By eliminating possible zeros and considering only the positive integers among r', s' ,and t' we have built a partition of $n-1$ into at most 3 parts.

In order to build the reverse map, be careful to add parts of size 1 to the partition of $\frac{m(n^2+n)}{2} + n+2$ when the original partition of $n-1$ has less than 3 parts.

Finally, the first equality of the theorem is easy to see since, by Proposition 4.2 again, any partition counted by the number $p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+n+2,3\right)$, or also by $p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+3+(n-1),3\right)$, has $n+j$ as its greatest part. □

Example 4.5. *For $m = 3$, $n = 5$ and $j = 4$, we have the partitions shown in Table 8.*

Table 8: Table for Example 4.25.

$P^{3[s]}_{[s]}(52,3)$	$P^{3[s]}_{[s]}(142,3)$	$P(4, \text{at most 3 parts})$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	(4)
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(3,1)$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(2,2)$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(9,9,9,\cdots,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(2,1,1)$

In order to have the 3rd diagonal of any table of function $f^m_*(q)$ completely described, for any $m \geq 2$, we need some simple identities, which we enunciate as a lemma.

Lemma 4.6.

(i) For all $j \geq 1$,

$$p(3j+1, \textit{exactly } 3 \textit{ parts}) = \frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor; \tag{4.1}$$

(ii) For all $j \geq 2$,

$$\sum_{i=1}^{j-1} \left\lfloor \frac{j+1-i}{2} \right\rfloor = \left\lfloor \frac{j^2}{4} \right\rfloor; \tag{4.2}$$

(iii) For all $j \geq 1$,

$$p(3j, \textit{exactly } 3 \textit{ parts}) = \left\lfloor \frac{j^2+1}{2} \right\rfloor + \left\lfloor \frac{j^2}{4} \right\rfloor. \tag{4.3}$$

Proof.

(i) According to [7], the number of partitions of n into exactly 3 parts is $\left\{ \frac{(n+3)^2}{12} \right\} - \left\lfloor \frac{n}{2} \right\rfloor - 1$. By writing $3j+1$ in place of n we get statement (4.1) by proving the following one:

$$\left\{ \frac{(3j+4)^2}{12} \right\} - \left\lfloor \frac{3j+1}{2} \right\rfloor - 1 = \frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor. \tag{4.4}$$

In order to prove (4.4) we write j in four different ways, according to its congruence modulus 4. First of all, note that

$$\left\{ \frac{(3j+4)^2}{12} \right\} = \left\{ \frac{9j^2+24j+16}{12} \right\} = \left\{ \frac{3j^2}{4} + 2j + \frac{4}{3} \right\} = 2j + \left\{ \frac{3j^2}{4} + \frac{4}{3} \right\},$$

and we may write (4.4) as

$$2j+\left\{\frac{3j^2}{4}+\frac{4}{3}\right\}-\left\lfloor\frac{3j+1}{2}\right\rfloor-1=\frac{j(j+1)}{2}+\left\lfloor\frac{j^2}{4}\right\rfloor. \tag{4.5}$$

We analyse separately each side of equation (4.5), and for any value of j modulus 4 we conclude that the equality is true.

(ii) We prove statement (4.2) by induction over j. Note that for $j=2$ we have

$$\sum_{i=1}^{2-1}\left\lfloor\frac{2+1-i}{2}\right\rfloor=\left\lfloor\frac{2+1-1}{2}\right\rfloor=1=\left\lfloor\frac{2^2}{4}\right\rfloor.$$

By supposing for certain $j=b\geq 2$ that we have

$$\sum_{i=1}^{b-1}\left\lfloor\frac{b+1-i}{2}\right\rfloor=\left\lfloor\frac{b^2}{4}\right\rfloor,$$

let us prove that

$$\sum_{i=1}^{(b+1)-1}\left\lfloor\frac{(b+1)+1-i}{2}\right\rfloor=\left\lfloor\frac{(b+1)^2}{4}\right\rfloor$$

or, which is the same, that

$$\sum_{i=1}^{b}\left\lfloor\frac{b+2-i}{2}\right\rfloor=\left\lfloor\frac{b^2+2b+1}{4}\right\rfloor. \tag{4.6}$$

First of all, note that if i and b have the same parity, then $\left\lfloor\frac{b+2-i}{2}\right\rfloor=\left\lfloor\frac{b+1-i}{2}\right\rfloor+1$.
And if i and b have different parities, then $\left\lfloor\frac{b+2-i}{2}\right\rfloor=\left\lfloor\frac{b+1-i}{2}\right\rfloor$.
If b is even, saying $b=2k$, then half of the values of i in the sum on the left hand side of (4.6) are even and the other half are odd. So we have

$$\sum_{i=1}^{b}\left\lfloor\frac{b+2-i}{2}\right\rfloor=\sum_{i=1}^{b}\left\lfloor\frac{b+1-i}{2}\right\rfloor+k=\sum_{i=1}^{b-1}\left\lfloor\frac{b+1-i}{2}\right\rfloor+k,$$

which by induction hypothesis equals

$$\left\lfloor\frac{b^2}{4}\right\rfloor+\frac{b}{2}=\left(\frac{2k}{2}\right)^2+\frac{2k}{2}=\frac{4k^2+4k}{4}=\left\lfloor\frac{b^2+2b}{4}\right\rfloor=\left\lfloor\frac{b^2+2b+1}{4}\right\rfloor,$$

as we wanted.

If b is odd, saying $b=2k+1$, then $k+1$ of the values of i in the sum on the left hand side of (4.6) are odd and k of those values are even. So we have

$$\sum_{i=1}^{b}\left\lfloor\frac{b+2-i}{2}\right\rfloor=\sum_{i=1}^{b}\left\lfloor\frac{b+1-i}{2}\right\rfloor+k+1=\sum_{i=1}^{b-1}\left\lfloor\frac{b+1-i}{2}\right\rfloor+k+1,$$

which by induction hypothesis equals

$$\begin{aligned}\left\lfloor\frac{b^2}{4}\right\rfloor+k+1 &= \left\lfloor\frac{4k^2+4k+1}{4}\right\rfloor+k+1=k^2+2k+1\\ &= \left(\frac{b-1}{2}\right)^2+b=\frac{b^2+2b+1}{4}=\left\lfloor\frac{b^2+2b+1}{4}\right\rfloor,\end{aligned}$$

as we needed.

So, by induction statement (4.2) is proved.

(iii) By using again the expression for the number of partitions of n into exactly 3 parts ([7]), with $3j$ in place of n we get what we need by proving the following statement:

$$\left\{\frac{(3j+3)^2}{12}\right\}-\left\lfloor\frac{3j}{2}\right\rfloor-1=\left\lfloor\frac{j^2+1}{2}\right\rfloor+\left\lfloor\frac{j^2}{4}\right\rfloor. \tag{4.7}$$

In order to prove (4.7) we write j in four different ways, according to its congruence modulus 4. First of all, note that

$$\left\{\frac{(3j+3)^2}{12}\right\}=\left\{\frac{9j^2+18j+9}{12}\right\}=\left\{\frac{3j^2+3}{4}+\frac{3j}{2}\right\}$$

and we rewrite (4.7) as

$$\left\{\frac{3j^2+3}{4}+\frac{3j}{2}\right\}-\left\lfloor\frac{3j}{2}\right\rfloor-1=\left\lfloor\frac{j^2+1}{2}\right\rfloor+\left\lfloor\frac{j^2}{4}\right\rfloor. \tag{4.8}$$

As done in the proof of item (i), we analyse separately each side of equation (4.8), and for any value of j modulus 4 we conclude that the equality is true. □

From Lemma 4.6 we finalize the characterization of the sequence of values $p_{[s]}^{m[s]}(n,3)$ by setting the three final theorems of this section.

By considering only the non-zero integers of the 3rd diagonal, the next theorem deals with the central terms of the lists of non-zero integers in even positions, that is, the central terms of the $2j$th lists of non-zero integers, for any $j\geq 1$.

Theorem 4.7. *Given $m\geq 3$, for all even $n\geq 2$, let us say $n=2j$ with $j\geq 1$, we have*

$$\begin{aligned}&p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-1,3\right)=T_j\\ &=p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-2,3\right).\end{aligned} \tag{4.9}$$

Before the proof, observe that, according to Proposition 4.2, for all $j \geq 1$ we have

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-1,3\right)=p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor,3\right)$$

and

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-2,3\right)=p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor+1,3\right).$$

So, Theorem 4.7 says that these four numbers are actually all the same and equal to the jth triangular number T_j. That is, the 3rd diagonal of the table of any function $f_*^m(q)$ has constant subsequences of size four located in lines

$$n=\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-2,\ n=\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor-1,$$

$$n=\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor,\ \text{and } n=\frac{m((2j)^2+2j)}{2}+3+\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor+1,$$

each of these four terms equal to T_j.

Proof of Theorem 4.7. Noting that $\left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor=3j-1$, we can rewrite statement (4.9) as

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3j+1,3\right)=T_j=p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3j,3\right).$$

According to Proposition 4.2, with $n=2j$ and respectively $i=3j-2$ and $i=3j-3$, the greatest part of any partition counted by

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3j+1,3\right)\ \text{and}\ p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2}+3j,3\right)$$

is $2j$. So, partitioning $\frac{m((2j)^2+2j)}{2}+3j+1$ and $\frac{m((2j)^2+2j)}{2}+3j$ according to our rules is the same as partitioning $3j+1$ and $3j$ into three parts less than or equal to $2j$. That is,

$$r_1+s_1+t_1=3j+1 \tag{4.10}$$

and

$$r_2+s_2+t_2=3j, \tag{4.11}$$

with $1\leq t_1\leq s_1\leq r_1\leq 2j$ and $1\leq t_2\leq s_2\leq r_2\leq 2j$.

Clearly, the number of solutions of equation (4.10) with no restriction on parts is precisely $p(3j+1, \textit{exactly}\ 3\ \textit{parts})$, which by item (i) of Lemma 4.6 is $\frac{j(j+1)}{2}+\left\lfloor\frac{j^2}{4}\right\rfloor$.

Now, we eliminate the solutions of $r_1+s_1+t_1=3j+1$ that do not satisfy $1\leq t_1\leq s_1\leq r_1\leq 2j$, which are those where $r_1>2j$, or $r_1=2j+i$ with $1\leq i\leq j-1$.

For each value of i, we have to eliminate the solutions of $s_1+t_1=j+1-i$, which we already know are in number of $\left\lfloor \dfrac{j+1-i}{2} \right\rfloor$. From item (ii) of Lemma 4.6, the total amount of solutions we have to eliminate is $\sum_{i=1}^{j-1} \left\lfloor \dfrac{j+1-i}{2} \right\rfloor = \left\lfloor \dfrac{j^2}{4} \right\rfloor$. Observe that item (ii) of Lemma 4.6 is valid only for $j \geq 2$. However, the case with $j=1$ has no solution to be eliminated, because $r_1 > 2$ never occurs in equation 4.10, since $r_1, s_1, t_1 \geq 1$.

Then, the number of solutions of $r_1+s_1+t_1=3j+1$ with the restriction $1 \leq t_1 \leq s_1 \leq r_1 \leq 2j$ is

$$\frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor - \left\lfloor \frac{j^2}{4} \right\rfloor = \frac{j(j+1)}{2} = T_j.$$

Left to the reader, equation (4.11) has an adaptable proof by using item (iii) of Lemma 4.6, and rearranging the indexes of the sum in item (ii). Therefore, equality (4.9) is proved and Theorem 4.7 is valid.

□

In an analogous way as done in Theorem 4.7, the next two theorems characterize the central terms of the lists of non-zero integers in odd positions. The proofs of both theorems use items (i) and (ii) of Lemma 4.6 with some replacements of j, and the formula for unrestricted partitions into 3 parts, available in [7], also necessary for proving items (i) and (iii) of Lemma 4.6.

Theorem 4.8. *Given $m \geq 3$, for all $n \equiv 1 \pmod 4$, let us say $n = 4j+1$ with $j \geq 0$, we have*

$$\begin{aligned} p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+3+\left\lfloor\frac{3\cdot(4j+1)-1}{2}\right\rfloor-1,3\right) &= j^2+(j+1)^2 \\ = p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+3+\left\lfloor\frac{3\cdot(4j+1)-1}{2}\right\rfloor-2,3\right)&+1. \end{aligned}$$

Theorem 4.9. *Given $m \geq 3$, for all $n \equiv 3 \pmod 4$, let us say $n = 4j+3$ with $j \geq 0$, we have*

$$\begin{aligned} p_{[s]}^{m[s]}\left(\frac{m((4j+3)^2+(4j+3))}{2}+3+\left\lfloor\frac{3\cdot(4j+3)-1}{2}\right\rfloor-1,3\right) &= 2(j+1)^2 \\ = p_{[s]}^{m[s]}\left(\frac{m((4j+3)^2+(4j+3))}{2}+3+\left\lfloor\frac{3\cdot(4j+3)-1}{2}\right\rfloor-2,3\right)&. \end{aligned}$$

5 THE 4th DIAGONAL

Most of the results from this section are similar to those presented in previous sections, as well as their proofs. Therefore we choose to exhibit only the proofs that are essentially different.

First of all, let us highlight in Table 9 some values of $p_{[s]}^{m[s]}(n,4)$ for $1 \leq n \leq 200$, contained in the 4th diagonal of the tables of functions $f_*^4(q)$, $f_*^5(q)$, and $f_*^6(q)$.

Table 9: Values contained in the 4th diagonals of the tables of functions $f_*^4(q)$, $f_*^5(q)$, and $f_*^6(q)$.

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,4)$
$f_*^4(q)$	$(0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0,1,1,2,2,3,2,2,1,1,0,0,$ $0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,0,0,1,1,2,3,5,5,7,7,8,7,7,5,5,$ $3,2,1,1,0,0,0,0,0,0,0,1,1,2,3,5,6,8,9,11,11,12,11,11,9,8,6,5,3,2,1,1,0,0,0,$ $0,0,0,0,1,1,2,3,5,6,9,10,13,14,16,16,18,16,16,14,13,10,9,6,5,3,2,1,1,0,0,0,$ $0,0,0,0,1,1,2,3,5,6,9,11,14,16,19,20,23,23,24,23,23,20,19,16,14,11,9,6,5,3,$ $2,1,1,0,0,0,0,0,0,0,1,1,2,3,5,6,9,11,15,17,21,23,27,28,31,31,33,31,31,28,$ $27,23,21,17,15,11,9,6,5,3,2,1,1,0,0,0,0,\ldots)$
$f_*^5(q)$	$(0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,$ $2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,$ $0,0,1,1,2,3,5,5,7,7,8,7,7,5,5,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,5,6,$ $8,9,11,11,12,11,11,9,8,6,5,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,5,6,$ $9,10,13,14,16,16,18,16,16,14,13,10,9,6,5,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,$ $0,0,1,1,2,3,5,6,9,11,14,16,19,20,23,23,24,23,23,20,19,16,14,11,9,6,5,3,2,1,$ $1,0,0,0,0,0,0,0,0,\ldots)$
$f_*^6(q)$	$(0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,$ $0,1,1,2,2,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,$ $0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,5,5,7,7,8,7,7,5,5,3,2,1,1,0,0,0,0,$ $0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,5,6,8,9,11,11,12,11,11,9,8,6,5,3,2,1,$ $1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,5,6,9,10,13,14,16,16,18,$ $16,16,14,13,10,9,6,5,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$ $0,1,\ldots)$

The following results and examples characterize the position of zeros and non-zero integers contained in the 4th diagonals, and also provide the exact values contained in those list of non-zero integers, based on a set of integer partitions easier to be counted.

Proposition 5.1. *Given $m \geq 4$, for all $n \geq 1$ and $1 \leq i \leq (m-4)n+7$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4-i,4\right)=0.$$

Non-zero lists of values in the 4th diagonal also obey a symmetry.

Proposition 5.2. *Given $m \geq 4$, for all $n \geq 1$ and $0 \leq i \leq 2n-2$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4+i,4\right)=p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4n-i,4\right).$$

Example 5.3. *For $m = 5$, $n = 4$ and $i = 5$, we have the same number of elements in $P_{[s]}^{4[s]}(59,4)$ as in $P_{[s]}^{4[s]}(61,4)$, as shown in Table 10.*

Table 10: Table for Example 5.32.

$P_{[s]}^{4[s]}(59,4)$	$P_{[s]}^{4[s]}(61,4)$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{3},\mathbf{1})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{4},\mathbf{2},\mathbf{1})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{2})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{2})$

The following theorem describes the identity in Proposition 5.2 in terms of a simpler type of partitions whose exact value is easier to find. Again its demonstration follows the lines of those analogous results from previous sections.

Theorem 5.4. *Given $m \geq 4$, for all $n \geq 1$ and $j \geq 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+3,4\right) = p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+n+3,4\right)$$
$$= p(n-1, \textit{at most } 4 \textit{ parts}).$$

Example 5.5. *For $m = 5$, $n = 6$ and $j = 4$, we have the partitions shown in Table 11.*

Table 11: Table for Example 5.34.

$P_{[s]}^{5[s]}(114,4)$	$P_{[s]}^{5[s]}(284,4)$	$P(5, \text{at most } 4 \text{ parts})$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	(5)
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})$	$(4,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(3,2)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(3,1,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(2,2,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(2,1,1,1)$

Recalling Remark 2.3, the notation $[n] := \{1,2,3,\ldots,n-1,n\}$ appears in the next theorem. It deals with the particular case of Proposition 5.2, with $i = 2n-2$, characterizing the central term of any list of non-zero integers in the sequence of values of $p_{[s]}^{m[s]}(n,4)$.

Theorem 5.6. *Given $m \geq 4$, for all $n \geq 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2n+2,4\right) = p(2n+8, \textit{exactly } 4 \textit{ distinct parts in } [n+3]).$$

Proof. First observe that $\dfrac{m(n^2+n)}{2}+2n+2=\dfrac{m(n^2+n)}{2}+4n-(2n-2)$.
Now, according to Proposition 5.2, the greatest part of any partition counted by the number $p_{[s]}^{m[s]}\left(\dfrac{m(n^2+n)}{2}+4n-(2n-2),4\right)$ is n. So we may write

$$r+s+t+u=2n+2,$$

with $1\leq u\leq t\leq s\leq r\leq n$.

By making

$$r'=r+3,$$
$$s'=s+2,$$
$$t'=t+1,$$

and

$$u'=u,$$

note that $1\leq u'<t'<s'<r'\leq n+3$. Therefore, r', s', t' and u' are distinct, they belong to $[n+3]$, and

$$\begin{aligned}r'+s'+t'+u' &= r+3+s+2+t+1+u\\ &= 2n+2+6\\ &= 2n+8.\end{aligned}$$

So we have $\mu=(r',s',t',u')\in P(2n+8,\text{exactly 4 distinct parts in }[n+3])$. The reverse map is simple to build. $\square$

Example 5.7. *For $m=4$ and $n=1,2,3$, we have the partitions shown in Table 12.*

Table 12: Table for Example 5.36.

$n=1$	$P_{[s]}^{4[s]}(8,4)$	$P(10,\text{exactly 4 distinct parts in }[4])$
	$(1,1,1,1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(4,3,2,1)$
$n=2$	$P_{[s]}^{4[s]}(18,4)$	$P(12,\text{exactly 4 distinct parts in }[5])$
	$(2,2,2,2,1,1,1,1,\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$(5,4,2,1)$
$n=3$	$P_{[s]}^{4[s]}(32,4)$	$P(14,\text{exactly 4 distinct parts in }[6])$
	$(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1})$ $(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{1})$ $(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$	$(6,5,2,1)$ $(6,4,3,1)$ $(5,4,3,2)$

6 SOME CONSIDERATIONS ABOUT THE *K*th DIAGONALS FOR $k \geq 5$

Some results of the previous sections involving partitions counted by $p_{[s]}^{m[s]}(n,k)$ were very similar for $k = 2,3,$ and 4. At this point of the text, we may already believe that those facts might be extensible for other values of k, or maybe even for any value of $k \geq 2$.

Indeed, a very simple fact that can be observed anywhere, in every table of functions $f_*^m(q)$, is that every kth diagonal seems to be formed by non-constant symmetrical list of integers, besides a few zeros between these lists. Both these results were presented in the previous sections of this chapter and can be generalized for any value of k, as it is set below. The proofs are similar to those from the previous sections and completely adaptable, thus some details are left to the reader.

Theorem 6.1. *Given $k \geq 2$ and $m \geq k$, for all $n \geq 1$ and $1 \leq i \leq (m-k)n+2k-1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k-i,k\right)=0.$$

Proof. Let us suppose, on the contrary, that $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k-i,k\right) \neq 0$ and so we could write

$$\begin{aligned}\frac{m(n^2+n)}{2}+k-i &= \underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{2+\cdots+2}_{m}+\underbrace{1+\ldots+1}_{m}+\lambda_1+\lambda_2+\cdots+\lambda_k\\ &= \frac{m(j^2+j)}{2}+\lambda_1+\cdots+\lambda_k,\end{aligned}$$

for some $j \geq 1$ and $1 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq j$. Note that $j < n$, and so we can make the following estimates:

$$\frac{m(n^2+n)}{2}+k-i=\frac{m(j^2+j)}{2}+\lambda_1+\lambda_2+\cdots+\lambda_k \leq \frac{m((n-1)^2+(n-1))}{2}+k(n-1),$$

which is equivalent to

$$mn+k-i \leq k(n-1).$$

On the other hand, as $i \leq (m-k)n+2k-1$, we have

$$mn+k-i \geq k(n-1)+1,$$

and so

$$k(n-1)+1 \leq k(n-1),$$

which is absurd.

As we cannot write $\frac{m(n^2+n)}{2}+k-i = \underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{1+\ldots+1}_{m}+\lambda_1+\lambda_2+\cdots+\lambda_k$, this means $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k-i,k\right)=0.$ □

Theorem 6.2. *Given $k \geq 3$ and $m \geq k$, for all $n \geq 1$ and $0 \leq i \leq \left\lfloor \frac{k(n-1)}{2} \right\rfloor$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k+i,k\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+kn-i,k\right).$$

Proof. By doing some estimates and using Lemma 3.2 we get that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k+i,k\right)$ or by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+kn-i,k\right)$ is exactly n.

Then, given λ a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k+i,k\right)$, we have

$$\lambda = (\underbrace{n,\ldots,n}_{m},\ldots,\underbrace{2,\ldots,2}_{m},\underbrace{1,\ldots,1}_{m},\lambda_1,\lambda_2,\ldots,\lambda_k),$$

with $1 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq n$. So,

$$m \cdot n + m \cdot (n-1) + \cdots + m \cdot 2 + m \cdot 1 + \lambda_1 + \lambda_2 + \cdots + \lambda_k = \frac{m(n^2+n)}{2} + k + i,$$

and therefore

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = k + i.$$

By writing $\mu = (\underbrace{n,\ldots,n}_{m},\ldots,\underbrace{2,\ldots,2}_{m},\underbrace{1,\ldots,1}_{m},n+1-\lambda_1,\ldots,n+1-\lambda_k)$ we get a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2k-i,k\right)$ as

$$\begin{aligned} m \cdot n + \cdots + m \cdot 2 + m \cdot 1 + n + 1 - \lambda_1 + \cdots + n + 1 - \lambda_k &= \frac{m(n^2+n)}{2} + kn + k - (\lambda_1 + \cdots + \lambda_k) \\ &= \frac{m(n^2+n)}{2} + kn + k - (k+i) \\ &= \frac{m(n^2+n)}{2} + kn - i. \end{aligned}$$

Easily we can build the reverse map, getting

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+k+i,k\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+kn-i,k\right).$$

□

7 CONCLUSIONS

According to the information contained in this work, in every table for any value of $m \geq 2$, the results we proved tell us exactly the number of partitions of n counted by $p_{[s]}^{m[s]}(n,k)$, for $k = 2,3,4$.

Moreover, the kth diagonal of any table generated by function $f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}$, for any k, alternates a sequence of zeros with a symmetrically increasing and decreasing sequence of non-vanishing numbers, which, in some particular cases, we may count in an easier way when seen as other type of partitions. The influence of m in such a sequence is restricted to the size of the sequence of zeros, while given a fixed k the non-vanishing sequences are the same in all tables.

Finally, we observe that every integer n is contemplated in some of the results of this paper. In particular, we may say that in some sense we have provided a complete characterization of a particular type of integer partition for every $n \in \mathbb{N}$.

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